J. DIFFERENTIAL GEOMETRY
22 (1985) 29–42

DECOMPOSITION THEOREMS FOR LORENTZIAN MANIFOLDS WITH NONPOSITIVE CURVATURE

JOHN K. BEEM, PAUL E. EHRLICH, STEEN MARKVORSEN
& GREGORY J. GALLOWAY

1. Introduction

The Toponogov Splitting Theorem [6] states that a complete Riemannian manifold (H, h) of nonnegative sectional curvatures which contains a line γ: $\mathbf{R} \to H$ (i.e., a complete absolutely minimizing geodesic) must be isometric to a product $\mathbf{R} \times H'$, the first factor being represented by γ. In [6] Cheeger and Gromoll gave a proof of this theorem stemming from their soul construction. Subsequently, Cheeger and Gromoll [5] were able to generalize this Riemannian splitting theorem to the case of nonnegative Ricci curvatures. In [17, p. 696], S. T. Yau raised the question of showing that a geodesically complete Lorentzian 4-manifold of nonnegative timelike Ricci curvature which contains a timelike line (i.e., a complete absolutely maximizing timelike geodesic) is isometrically the Cartesian product of that geodesic and a spacelike hypersurface.

Galloway [9] has recently considered this question for space-times which are spatially closed, i.e., which admit a smooth time function whose level sets are compact (smooth) Cauchy surfaces. Let (M, g) be such a globally hyperbolic space-time which satisfies the strong energy condition $\mathrm{Ric}(v, v) \geqslant 0$ for all timelike vectors v in TM. Suppose further that (M, g) contains a timelike curve which is both future and past complete and that for each $p \in M$, every null geodesic emanating from p contains a past and future null cut point to p.

Received August 25, 1984 and, in revised form, June 29, 1985. This material is based upon work of the first and second authors which was supported by the National Science Foundation under grant number DMS-840 1599 and was also supported in part by a grant from the Weldon Spring fund of the University of Missouri. The third author was supported in part by the Danish Natural Science Research Council.

Then Galloway shows (M, g) splits isometrically as a Lorentzian product $(\mathbf{R} \times H, -dt^2 \otimes h)$, where (H, h) is a compact Riemannian manifold. The proof employs and extends some results of [1] and [10].

In the present paper, we consider a different class of space-times than those studied in [9] and we use quite different techniques to obtain the following splitting theorem.

Theorem 5.2. *Let (M, g) be a globally hyperbolic space-time of dimension $\geqslant 2$ with everywhere nonpositive timelike sectional curvatures $K \leqslant 0$ which contains a complete timelike line γ: $(-\infty, \infty) \to (M, g)$. Then (M, g) is isometric to a product $(\mathbf{R} \times H, -dt^2 \otimes h)$, where (H, h) is a complete Riemannian manifold. The factor $(\mathbf{R}, -dt^2)$ is represented by γ and (H, h) is represented by a level set of a Busemann function associated to γ.*

This theorem provides an affirmative answer to the question raised by Yau for globally hyperbolic space-times with nonpositive timelike sectional curvatures without imposing the assumption of geodesic completeness.

We work directly with the Busemann function in obtaining Theorem 5.2 as in [5] rather than dealing with direct geometric constructions as in the Riemannian proof in [6]. We have also been influenced by a series of papers by R. Greene and H. Wu (cf. [16] for a survey) and by a paper of Eschenburg and Heintze [7].

We would like to thank J.-H. Eschenburg and E. Heintze for providing us with a preprint of [7].

2. Preliminaries

In this paper (M, g) will always be a connected, time oriented Lorentzian manifold which is globally hyperbolic with metric g of signature $(-, +, \cdots, +)$. If A is a subset of M, then $I^+(A) = \{q \in M \,|\, a \ll q \text{ for some } a \in A\}$ and $I^-(A)$ is defined dually. The sets $I^+(p)$, $I^-(p)$, $I^+(A)$, and $I^-(A)$ are always open. Furthermore, we set $I(A) = I^+(A) \cap I^-(A)$.

Given $p, q \in M$, set $d(p, q) = 0$ if $q \notin J^+(p)$ and let $d(p, q)$ be the supremum of lengths of future directed causal curves from p to q if $q \in J^+(p)$. The *Lorentzian distance function* d satisfies the *reverse triangle inequality* $d(p, q) \geqslant d(p, r) + d(r, q)$ whenever $p \leqslant r \leqslant q$. Since (M, g) is globally hyperbolic, the Lorentzian distance function is both finite valued and continuous, (cf. [13]).

A causal geodesic is *maximal* if the length between any pair of its points is equal to the Lorentzian distance between these points. For a unit speed future directed timelike geodesic $\gamma\colon (a,b) \to M$, this means that $d(\gamma(t_1), \gamma(t_2)) = t_2 - t_1$ for all $a < t_1 < t_2 < b$. A maximal timelike geodesic γ is a *line* if it is complete (i.e., $a = -\infty$ and $b = \infty$). A maximal causal geodesic of the form $\gamma\colon [a, \infty) \to M$ is called a *causal ray*.

Most of our notational conventions are standard and may be found in [2], [13], and [14].

3. Busemann functions

If $\gamma\colon (-\infty, \infty) \to M$ is a future directed timelike line, then for each fixed $r \geqslant 0$ we define $b_r^+\colon M \to \mathbf{R}$ by $b_r^+(x) = r - d(x, \gamma(r))$. (cf. [3, p. 131], [5, p. 119]). These functions are continuous functions of both x and r because (M, g) is globally hyperbolic.

If $\gamma(r) \notin I^+(x)$, then $d(x, \gamma(r)) = 0$ and $b_r^+(x) = r$. Thus $b_r^+(x)$ is an increasing function of r for fixed x as long as $\gamma(r) \notin I^+(x)$. On the other hand, if $x \ll \gamma(r)$ for some $r \geqslant 0$, then there is a smallest $r_0 \geqslant 0$ such that $x \ll \gamma(r)$ for all $r_0 < r < \infty$. Assuming such an r_0 exists, the reverse triangle inequality implies that $b_r^+(x)$ is a monotone decreasing function of r for all $r > r_0$. If we then allow the possible values of $-\infty$ and $+\infty$, the *Busemann function*

$$b^+(x) = \lim_{r \to \infty} b_r^+(x)$$

exists for all $x \in M$. In the case $x \notin I^-(\gamma)$ we have $b^+(x) = +\infty$. In general, b^+ need not be a continuous function of x for globally hyperbolic space-times. In fact, examples conformal to a subset of the Minkowski plane L^2 may be constructed with b^+ discontinuous.

If $x \ll \gamma(r)$ for all $r_0 < r < \infty$, given a sequence of points $\{x_n\}$ converging to x and a sequence of numbers $\{r_n\}$ diverging to $+\infty$ we will have $x_n \ll \gamma(r_n)$ for all sufficiently large n by the openness of chronological sets. We will implicitly use the properties of limit curves (cf. [2]) to define the notion of co-ray to γ as follows. A (future) *co-ray* to γ from x will be a causal curve starting at x which is future inextendible and is the limit curve of a sequence of maximal length timelike geodesic segments from x_n to $\gamma(r_n)$ for two sequences $\{x_n\}$, $\{r_n\}$ with $x_n \to x$ and $r_n \to \infty$ (cf. [2, pp. 33–45]). Past co-rays are defined dually. This definition of co-ray corresponds to that used by Busemann [3, p. 130]. A (future) co-ray is always a maximal length causal geodesic starting at a point x. However, there may be more than one co-ray to γ from x. Furthermore, a co-ray to γ from x may be a null geodesic. To rule out this possibility, we impose the following condition on (M, g).

Definition 3.1. The globally hyperbolic space-time (M, g) satisfies the *timelike co-ray condition* for the timelike line $\gamma\colon (-\infty, \infty) \to M$ if for each $x \in I^+(\gamma) \cup I^-(\gamma)$ all future and past co-rays to γ from x are timelike.

The timelike co-ray condition has the following technical consequences which may be obtained using standard arguments.

Lemma 3.2. *Let (M, g) be a globally hyperbolic space-time which satisfies the timelike co-ray condition for the timelike line $\gamma\colon (-\infty, \infty) \to (M, g)$. Let $x \in I^-(\gamma)$ be arbitrary and $\varepsilon > 0$ be given. Then there exists an integer $N > 0$, a neighborhood $U(x)$ of x with $U(x) \subset I^-(\gamma)$ and an open set V with compact closure $K = \overline{V} \subset I^-(x)$ satisfying the following properties*:

(a) *$K \subset I^+(y)$ for all $y \in U(x)$.*

(b) *Given any $y \in U(x)$ and $q \in K$, we have $d(y, q) < \varepsilon$.*

(c) *If $y \in U(x)$ and $t \geqslant N$, then any maximal timelike geodesic segment from y to $\gamma(t)$ intersects K.*

We now show b^+ is continuous on the set $I^-(\gamma)$.

Lemma 3.3. *Let (M, g) be globally hyperbolic. If (M, g) satisfies the timelike co-ray condition for the timelike line $\gamma\colon (-\infty, \infty) \to M$, then the Busemann function b^+ is continuous and finite on $I^-(\gamma)$.*

Proof. Let $\varepsilon > 0$ be given and fix $x \in I^-(\gamma)$. Let $U(x)$, N and K be as in Lemma 3.2. Choose any two points $y_1, y_2 \in U(x)$ and any r with $N < r < \infty$. Let G_i be a maximal length timelike geodesic from y_i to $\gamma(r)$ and let $q_i \in G_i \cap K$ for $i = 1, 2$. Then $d(y_1, \gamma(r)) = d(y_1, q_1) + d(q_1, \gamma(r))$ and $d(y_1, q_1) < \varepsilon$ yield $d(q_1, \gamma(r)) > d(y_1, \gamma(r)) - \varepsilon$. Since $y_2 \ll q_1 \ll \gamma(r)$ by Lemma 3.2(a), the reverse triangle inequality yields

$$d(y_2, \gamma(r)) \geqslant d(y_2, q_1) + d(q_1, \gamma(r)).$$

Thus

$$d(y_2, \gamma(r)) > 0 + d(y_1, \gamma(r)) - \varepsilon$$

which implies $\varepsilon > b_r^+(y_2) - b_r^+(y_1)$. Since also $q_2 \in I^+(y_1)$, we may reverse the roles of y_1 and y_2 to obtain

$$|b_r^+(y_1) + b_r^+(y_2)| < \varepsilon.$$

This establishes the equicontinuity of the functions b_r^+ on $U(x)$ for all $N < r < \infty$. Since $b_r^+(x)$ is monotone decreasing for large r, the limit $b^+(x)$ is an element of $\mathbf{R} \cup \{-\infty\}$. If $b^+(x) = -\infty$, then $b_r^+(x) \to -\infty$ as $r \to \infty$ and the last inequality yields $b^+(y) = -\infty$ for all $y \in U(x)$. It follows that b^+ equals

$-\infty$ on an open subset V_1 of $I^-(\gamma)$. On the other hand, if $b^+(x)$ is finite, then the last inequality yields $|b^+(x) - b^+(y)| \leqslant \varepsilon$ for all $y \in U(x)$. In this case the equicontinuous family $\{b_r^+\}$ for $N < r < \infty$ is bounded on $U(x)$ and hence converges to a finite valued continuous function b^+ on $U(x)$. Thus b^+ is a continuous finite valued function on an open subset V_2 of $I^-(\gamma)$ and is equal to $-\infty$ on the open subset V_1 of $I^-(\gamma)$, where $I^-(\gamma) = V_1 \cup V_2$. Using $b^+(\gamma(r_1)) = r_1$ on γ and the connectedness of $I^-(\gamma)$ it follows that $V_2 = I^-(\gamma)$ which establishes the lemma. q.e.d.

Since $\gamma\colon (-\infty, \infty) \to M$ is a timelike *line* rather than just a ray, one may also define a Busemann function b^- on $I^+(\gamma)$ by

$$b_r^-(x) = r - d(\gamma(-r), x), \qquad r \geqslant 0.$$

$$b^-(x) = \lim_{r \to +\infty} b_r^-(x).$$

Arguments similar to Lemma 3.3 imply that b is continuous on $I^+(\gamma)$. Thus both b^+ and b^- are continuous on $I(\gamma) = I^+(\gamma) \cap I^-(\gamma)$. Given any $x \in I(\gamma)$, choose positive numbers s and t such that $\gamma(-s) \ll x \ll \gamma(t)$. The reverse triangle inequality and $d(\gamma(-s), \gamma(t)) = s + t$ yield $b_t^+(x) + b_s^-(x) \geqslant 0$. Letting $s, t \to \infty$ we obtain the following inequality on $I(\gamma)$:

$$B(x) := b^+(x) + b^-(x) \geqslant 0. \tag{3.1}$$

4. The significance of nonpositive timelike sectional curvature

The two-plane $E = \{u, v\}$ is *timelike* if the metric induced on E is Lorentzian. Thus (M, g) has nonpositive timelike sectional curvatures if for each timelike plane E we have $\langle R(u, v)v, u\rangle \geqslant 0$.

Proposition 4.1. *Let (M, g) be a globally hyperbolic space-time of dimension $\geqslant 2$ with everywhere nonpositive timelike sectional curvatures $K \leqslant 0$ which contains a complete timelike line γ. Then the timelike co-ray condition holds on $I(\gamma)$. Thus b^+ and b^- are both continuous on $I(\gamma)$.*

Proof. Assume that $x \in I(\gamma)$ has a future co-ray σ to γ such that σ is null. Then there exist sequences $\{x_n\}$, $\{r_n\}$ and $\{\sigma_n\}$ with $x_n \to x$, $r_n \to \infty$ such that each σ_n is a maximal timelike geodesic segment from x_n to $\gamma(r_n)$ and σ is a limit curve of the sequence $\{\sigma_n\}$. Choose $q \in \gamma \cap I^-(x)$ and let μ_n be a maximal timelike geodesic segment from q to x_n. The segments μ_n are defined for all large n and we may assume that $\{\mu_n\}$ converges to a *timelike* geodesic from q to x. Let γ_n be the segment of γ from q to $\gamma(r_n)$ and set $a_n = L(\mu_n)$, $b_n = L(\sigma_n)$ and $c_n = L(\gamma_n)$, where L denotes arc length. Assuming μ_n, σ_n and

γ_n are parametrized by arclength, define $\beta_n = g(\mu'_n(0), \gamma'_n(0))$ and $\theta_n = g(-\mu'_n(a_n), \sigma'_n(0))$. Then $a_n \to a = d(q, x)$ and $\beta_n \to \beta = g(\mu'(0), \gamma'(0))$. We now apply Harris' triangle comparison theorem (cf. [2, p. 430], [11, p. 303]) to the timelike geodesic triangle $(\mu_n, \sigma_n, \gamma_n)$ using the two-dimensional Minkowski plane as model space. Thus for each n there is a timelike geodesic triangle $(\bar{\mu}_n, \bar{\sigma}_n, \bar{\gamma}_n)$ in L^2 with

$$a_n = L(\bar{\mu}_n), \quad b_n = L(\bar{\sigma}_n), \quad c_n = L(\bar{\gamma}_n), \quad |\bar{\beta}_n| \leqslant |\beta_n|, \quad \bar{\theta}_n \geqslant \theta_n,$$

where $\bar{\beta}_n$ and $\bar{\theta}_n$ are defined analogously to β_n and θ_n. Using $|\bar{\beta}_n| \leqslant |\beta_n|$ and $\beta_n \to \beta$, we find there is some positive C with $|\bar{\beta}_n| \leqslant C$ for all n. The law of cosines for L^2 yields

$$c_n^2 = a_n^2 + b_n^2 + 2a_n b_n \bar{\theta}_n, \tag{4.1}$$

$$b_n^2 = a_n^2 + c_n^2 + 2a_n c_n \bar{\beta}_n. \tag{4.2}$$

Adding (4.2) to (4.1) and solving for $\bar{\theta}_n$ we obtain

$$\bar{\theta}_n = -\frac{c_n \bar{\beta}_n}{b_n} - \frac{a_n}{b_n}.$$

Since $a_n \to a$, $c_n \to \infty$ and $|\bar{\beta}_n| \leqslant C$, equation (4.2) implies $b_n/c_n \to 1$ and $b_n \to \infty$. Consequently, there is some constant C_1 such that $\bar{\theta}_n \leqslant C_1$ for all n. Using $\theta_n \leqslant \bar{\theta}_n$ we find

$$g\big(-\mu'_n(a_n), \sigma'_n(0)\big) \leqslant C_1$$

which implies that the Lorentzian angles between the segments μ_n and σ_n are all bounded by some constant. On the other hand, since σ is assumed to be null, then $\mu_n \to \mu$ and $\sigma_n \to \sigma$ imply these angles must approach ∞. This contradiction establishes the result. q.e.d.

We now show that co-rays to γ are complete given $K \leqslant 0$.

Lemma 4.2. *Let (M, g) be a globally hyperbolic space-time with everywhere nonpositive timelike sectional curvatures $K \leqslant 0$ which contains a complete timelike line γ. If η is a future (resp. past) directed timelike co-ray from $x \in I(\gamma)$, then η is future (resp. past) complete.*

Proof. Assume w log η is future directed. Suppose η has finite length L. Let γ: $(-\infty, \infty) \to M$ and η: $[0, L) \to M$ be parametrized with respect to arclength.

By Lemma 3.2 there exists a constant $K > 0$ and "time" T such that if $t > T$ and μ is a maximal segment from x to $\gamma(t)$, then

$$\big|\big\langle \mu'(0), \eta'(0)\big\rangle\big| \leqslant K.$$

Also, by the reverse triangle inequality,

$$d(x,\gamma(t)) \to +\infty \quad \text{as } t \to +\infty.$$

Thus, we can choose $t_0 \geqslant T$ such that

$$d(x,\gamma(t)) \geqslant 3LK \quad \text{for } t \geqslant t_0.$$

Pick a point $y \in \eta$ such that $\gamma(t_0) \notin I^+(y)$. (If no such point existed, then η would be imprisoned in $J^-(\gamma(t_0)) \cap J^+(x)$.) Choose y_1 on η with $y_1 \in I^+(y)$. Since η is a co-ray to γ there exist points $p_n \to y_1$ such that each p_n lies on some maximal timelike geodesic from x to a point q_n of γ. It follows that for sufficiently large n we have $p_n \in I^+(y)$ which implies $q_n \in I^+(y)$. Consequently, there exists a time $t_1 > t_0$ such that $\gamma(t_1) \in \partial I^+(y)$. By the global hyperbolicity of M, $\partial I^+(y) = J^+(y) - I^+(y)$, and hence $d(y,\gamma(t_1)) = 0$. Thus one can choose a time $t_2 > t_1$ such that

$$0 < d(y,\gamma(t_2)) < 3LK/2$$

since the Lorentzian distance function is continuous for globally hyperbolic space-times. Let μ be a maximal segment from x to $\gamma(t_2)$ and let σ be a maximal segment from y to $\gamma(t_2)$. Let ν be the portion of η from x to y. Consider the timelike triangle (ν,μ,σ). Let a, b and c be the lengths of ν, σ and μ, respectively. We have

(1) $c \geqslant 3LK$, $b < c/2$ and $a < L$.

Also, $\beta = \langle \mu'(0), \eta'(0)\rangle$ satisfies

(2) $|\beta| \leqslant K$.

By Harris' triangle comparison theorem there exists a corresponding timelike triangle $(\bar{\mu}, \bar{\mu}, \bar{\sigma})$ in Minkowski space such that $L(\bar{\nu}) = a$, $L(\bar{\mu}) = b$, $L(\bar{\sigma}) = c$ and $|\bar{\beta}| \leqslant |\beta|$. By the law of cosines,

$$b^2 = a^2 + c^2 - 2ac|\bar{\beta}|.$$

Using (1), (2) and $|\bar{\beta}| \leqslant |\beta|$, we obtain

$$b^2 \geqslant c^2 - 2LKc = c^2\left(1 - \frac{2LK}{c}\right) \geqslant \frac{c^2}{3}$$

which contradicts the second inequality in (1). Thus η must be infinite in length. q.e.d.

At this juncture, we do *not* know that the functions B, $b^=$ and b^- defined in §3 are differentiable functions. Thus as in [7], we now need to define smooth local support functions at each $p \in I(\gamma)$ for b^+ and b^-. Fix $p \in I(\gamma)$ and a sequence of real numbers $\{r_n\}$ with $r_n \to +\infty$. Set $\alpha_n = d(p, \gamma(r_n))$. Then α_n is positive for all sufficiently large n, $\alpha_n \to \infty$ by the reverse triangle inequality, and

$$b^+(p) = \lim_{n\to\infty} (r_n - \alpha_n). \tag{4.3}$$

Furthermore, for each sufficiently large n there is some unit future directed timelike vector $v_n^+ \in T_pM$ such that $\exp_p(\alpha_n v_n^+) = \gamma(r_n)$. Using the timelike co-ray condition, we may assume that $v_n^+ \to v^+$, where v^+ is a unit future directed timelike vector at p. The future inextendible timelike geodesic with initial velocity vector v^+ is a co-ray to γ from p and is future complete by Lemma 4.2. Choose a sufficiently small neighborhood $U(p)$ of p, such that $x \ll \gamma(r_n)$ for all n larger than some N and all $x \in U(p)$. Fixing $U(p)$ and N there is some $0 < a_0 < \alpha_n$ such that

$$x \ll \exp_p(a_0 v_n^+) \ll \exp_P(a v_n^+) \ll \exp_p(\alpha_n v_n^+) = \gamma(r_n)$$

for all $n \geq N$, each a with $a_0 < a < \alpha_n$, and all $x \in U(p)$. Applying the reverse triangle inequality yields

$$\begin{aligned} d(x, \gamma(r_n)) &\geq d\big(x, \exp_p(a v_n^+)\big) + d\big(\exp_p(a v_n^+), \gamma(r_n)\big) \\ &\geq d\big(x, \exp_p(a v_n^+)\big) + \alpha_n - a. \end{aligned}$$

We obtain $r_n - \alpha_n + a - d(x, \exp_p(a v_n^+)) \geq r_n - d(x, \gamma(r_n))$. In view of equation (4.3), we thus have

$$b^+(p) + a - d\big(x, \exp_p(a v^+)\big) \geq b^+(x) \tag{4.4}$$

for all $x \in U(p)$ and $a_0 < a < \infty$. This inequality motivates the following definition of a family of functions $b_{p,a}^+$ (cf. [7]):

$$b_{p,a}^+(x) = b^+(p) + a - d\big(x, \exp_p a v^+\big). \tag{4.5}$$

Equation (4.4) shows that for sufficiently large a these functions are continuous *local super support functions* for the Busemann function b^+; we have $b_{p,a}^+(p) = b^+(p)$ and $b_{p,a}^+(x) \geq b^+(x)$ for all x sufficiently close to p.

Now given $p \in I(\gamma)$, construct a unit past directed timelike vector $v^- \in T_pM$ using the same technique as for v^+. Then

$$b_{p,a}^-(x) = b^-(p) + a - d\big(\exp_p(a v^-), x\big) \tag{4.6}$$

provides a family of local super support functions for b^-. That is, $b_{p,a}^-(p) = b^-(p)$ and $b_{p,a}^-(x) \geq b^-(x)$ for all x near p and large a for any $p \in I(\gamma)$. For any fixed parameter value a, the nonspacelike cut locus of $\exp_p(a v^+)$ is closed (cf. [2, p. 242]) and since $s \to \exp_p(s v^\pm)$, $s \in [0, a]$ is maximal, we get that there is a neighborhood of p in which $b_{p,a}^+$ (resp., $b_{p,a}^-$) is a *smooth* super support function for the continuous Busemann function b^+ (resp., b^-). Hence $B_{p,a} = b^+_{p,a} + b^-_{p,a}$ is also a smooth super support function for $B = b^+ + b^-$ near p.

In view of the definition of $b^+(x)$ and $b^-(x)$, we now consider functions of the form $f(x) = d(q,x)$ (resp., $d(x,q)$), where q is a given point of M. In general, these functions will fail to be differentiable across null cones as well as on the timelike cut locus of q (cf. [2, p. 105]). Since the null cut locus and nonspacelike cut locus of each $q \in M$ are closed, if $q \ll p$ and p is not in the (future) cut locus of q, then the function $f(x) = d(q,x)$ is smooth on some neighborhood $U(p)$ of p which contains no cut points of q. Furthermore, $\langle \operatorname{grad} f, \operatorname{grad} f \rangle = -1$ on $U(p)$. Conversely, let $f\colon M \to \mathbf{R}$ be a smooth function on an arbitrary Lorentzian manifold (M,g) with $\langle \operatorname{grad} f, \operatorname{grad} f \rangle \equiv -1$. Using the definition of Lorentzian arc length and the reverse Schwartz inequality it is easy to show that any integral curve c of $\operatorname{grad} f$ is a maximal timelike geodesic.

Lemma 4.3. *Let N be an open subset of the Lorentzian manifold (M,g) and assume $f\colon N \to \mathbf{R}$ is a smooth function with $\langle \operatorname{grad} f, \operatorname{grad} f \rangle \equiv -1$ on N. Let $c\colon (a,b) \to N$ be an integral curve of $-\operatorname{grad} f$ and V be any unit parallel vector field which is orthogonal to c. Then*

$$-\langle R(V,c')c', V\rangle \geqslant -(\operatorname{Hess}(f)(V,V)\circ c)' + (\operatorname{Hess}(f)(V,V)\circ c)^2.$$

Let W be an arbitrary parallel field along c. Set

$$\alpha = \alpha(W,c') = \sqrt{\langle W,W\rangle + (\langle W,c'\rangle)^2}.$$

If W is not proportional to c', set

$$V_1 = W + \langle W,c'\rangle c' \quad \text{and} \quad V = V_1/\|V_1\|. \tag{4.7}$$

Then $\alpha V = W + \langle W,c'\rangle c'$ and V is a spacelike unit parallel field which is orthogonal to c. Since $\langle \operatorname{grad} f, \operatorname{grad} f \rangle = -1$, one also obtains $\operatorname{Hess}(f)(X,c') = 0$ for any vector field X along c. Thus with α and V as above

$$\operatorname{Hess}(f)(W,W) = \alpha^2 \operatorname{Hess}(f)(V,V) \tag{4.8}$$

and we obtain the following estimate on the Hessian of $d(q,x)$.

Proposition 4.4. *Let (M,g) be a globally hyperbolic space-time of everywhere nonpositive timelike sectional curvature. Fix $p \in M$ and let $q \in I^+(p) \cup I^-(p)$ be any point which is not a cut point of p. Let $c\colon [0,L] \to M$ denote a unit speed maximal timelike geodesic from q to p. If $q \in I^+(p)$, set $f(x) = d(x,q)$*

and if $q \in I^-(p)$, set $f(x) = d(q,x)$. Then

$$-\frac{\alpha^2(w)}{f(p)} \leqslant \mathrm{Hess}(f)(w,w)|_p \tag{4.9}$$

for any $w \in T_pM$, where $\alpha^2(w) = \langle w,w\rangle + (\langle w, c'(L)\rangle)^2$.

Proof. As the proofs for $d(x,q)$ and $d(q,x)$ are dual, we only give the proof for $f(x) = d(q,x)$ and $q \in I^-(p)$. Let N be an open subset of $I^+(q)$ such that $c(s) \in N$ for all $0 < s \leqslant d(q,p)$ and such that N contains no cut points of q. Then c is an integral curve of $-\mathrm{grad}\, f$ for all $0 < s \leqslant d(q,p)$ and f is C^∞ on N. Given a fixed $w \in T_pM$ let V be the unit parallel field along c which satisfies $\alpha V = w + \langle w, c'\rangle c'$ as in equation (4.7). Define $\theta(s) = \mathrm{Hess}(f)(V,V) \circ c(s)$ for all $0 < s \leqslant d(q,p)$. At $x = c(s)$ we have $\mathrm{Hess}(f)(V,V)|_x = -\langle \nabla_V c', V\rangle|_x = S_{c'}(V,V)|_x$, where $S_{c'}$ is the second fundamental form of the distance sphere $\{y \in M \mid d(q,y) = d(q,x)\}$ through x. Thus $\mathrm{Hess}(f)(V,V)|_{c(s)} \to -\infty$ as $s \to 0^+$.

Lemma 4.3 and the curvature assumption yield $\theta^2 - \theta' \leqslant 0$ which implies that θ is nondecreasing and for all $\theta(s) \neq 0$ that

$$\theta^{-1}(s) \leqslant -s \tag{4.10}$$

using $d/ds(\theta^{-1}) = -\theta'/\theta^2$ and $\theta(s) \to -\infty$ as $s \to 0^+$. Thus for all $s > 0$, we have $\theta(s) \geqslant s^{-1}$. Setting $s = d(q,p) = f(q)$ yields

$$\mathrm{Hess}(f)(V,V)|_p \geqslant -\frac{1}{f(p)}. \tag{4.11}$$

The result now follows using equations (4.8) and (4.11).

Corollary 4.5. *Let (M,g) be a globally hyperbolic space-time with $K \leqslant 0$ and suppose that (M,g) contains a timelike line $\gamma\colon (-\infty,\infty) \to (M,g)$. Then for any $p \in I(\gamma)$ and $a > 0$, we have*

$$\mathrm{Hess}(b^+_{p,a})(w,w) \leqslant \frac{\alpha^2_+(w)}{a}, \tag{4.12}$$

$$\mathrm{Hess}(b^-_{p,a})(w,w) \leqslant \frac{\alpha^2_-(w)}{a} \tag{4.13}$$

for any $w \in T_pM$, where $\alpha^2_+(w) = \langle w,w\rangle + (\langle w, v^+\rangle)^2$ and $\alpha^2_-(w) = \langle w,w\rangle + (\langle w, v^-\rangle)^2$.

Proof. Since the arguments for (4.12) and (4.13) are similar, it suffices to establish (4.12). Consider the function $f(x) = d(x,q)$ with $q := \exp_p(av^+)$. Since γ satisfies the timelike co-ray condition, $c^+(t) = \exp_p(tv^+)$, $t \geqslant 0$, is a maximal, future directed, future complete timelike geodesic ray. Hence for any $a > 0$, $q = \exp_p(av^+) \in I^+(p)$ is not a cut point to p and $d(p,q) = a$. Thus inequality (4.9) may be applied to $f(x) = d(x,q)$ at $x = p$ to yield inequality (4.12) as $c'(L) = -v^+$. q.e.d.

Using the timelike co-ray condition, Corollary 4.5 and a one-dimensional Calabi-type maximum principle argument, we now show that the function B defined by equation (3.1) vanishes on $I(\gamma)$.

Lemma 4.6. *If $y \in I(\gamma)$ and $B(y) = 0$, then B vanishes on a neighborhood of y. Hence $B \equiv 0$ on $I(\gamma)$.*

Proof. If the conclusion fails, there is some geodesic segment σ: $[-1, 1] \to I(\gamma)$ of g with $\sigma(0) = y$ and $B(\sigma(s_1)) > 0$, where $0 < s_1 < 1$. Let h: $[-s_1, s_1] \to \mathbf{R}$ be defined by $h(s) = -\varepsilon(s + s^2)$. If $\varepsilon > 0$ is chosen sufficiently small, the continuous function $B \circ \sigma + h$ is positive at both endpoints of $[-s_1, s_1]$ and zero at $s = 0$. Thus $B \circ \sigma + h$ attains a minimum at some $s_0 \in (-s_1, s_1)$. Let $p = \sigma(s_0)$ and choose a future (resp. past) timelike vector v^+ (resp. v^-) at p tangent to a co-ray to γ from p. Using equations (4.5) and (4.6) define the local support functions $b^+_{p,a}$ (resp., $b^-_{p,a}$) using v^+ (resp., v^-). Then the function $B_{p,a}(x) = b^+_{p,a}(x) + b^-_{p,a}(x)$ is smooth and satisfies $B_{p,a}(p) = B(p)$ and $B_{p,a}(x) \geqslant B(x)$ in some neighborhood $U(p)$ of p. Applying Corollary 4.5 with $w = \sigma'(s_0)$ we obtain

$$\left(B_{p,a} \circ \sigma\right)''\Big|_p \leqslant \left(\alpha_+^2(w) + \alpha_-^2(w)\right)/a.$$

Thus

$$\left(B_{p,a} \circ \sigma + h\right)''\Big|_p \leqslant \left(\alpha_+^2(w) + \alpha_-^2(w)\right)a^{-1} - 2\varepsilon < 0$$

for sufficiently large a. This contradicts

$$B_{p,a} \circ \sigma(s) + h(s) \geqslant B \circ \sigma(s) + h(s) \geqslant B_{p,a}(p) + h(s_0)$$

for all $-s_1 \leqslant s \leqslant s_1$ with $\sigma(s) \in U(p)$. q.e.d.

We have been unable to extend Lemma 4.6 with our proof method to the case of the strong energy condition $\mathrm{Ric}(v, v) \geqslant 0$ for all timelike vectors v because the d'Alembertian operator (which corresponds to the Laplacian in the Riemannian case) is hyperbolic rather than elliptic.

Lemma 4.6 implies $b^+(x) \equiv -b^-(x)$ on $I(\gamma)$.

Lemma 4.7. *The Busemann functions b^+ and b^- are once differentiable on $I(\gamma)$ and the vector field $V = \operatorname{grad} b^+ = -\operatorname{grad} B^-$ is a unit past directed timelike vector field defined on $I(\gamma)$. The vector field V is continuous and at each point $p \in I(\gamma)$ there is a unique future directed co-ray $c^+(t) = \exp_p(-tV)$ and a unique past directed co-ray $c^-(t) = \exp_p(tV)$ to γ. These co-rays to γ at p fit together to form a (distance realizing and complete) timelike line.*

Proof. Choose $p \in I(\gamma)$ and let v^+ and v^- be unit timelike vectors at p which determine future and past directed co-rays at p, respectively. Let $b^+_{p,a}$ and $b^-_{p,a}$ be defined using v^+ and v^-, respectively according to equations (4.5) and (4.6). Now $B_{p,a}(x) = b^+_{p,a}(x) + b^-_{p,a}(x) \geqslant B(x) = 0$ and $B_{p,a}(p) = B(p) = 0$ imply the smooth function $B_{p,a}(x)$ satisfies $\operatorname{grad} B_{p,a}|_p = 0$. Thus $v^+ = -\operatorname{grad} b^+_{p,a}|_p = \operatorname{grad} b^-_{p,a}|_p = -v^-$ so that $v^+ = -v^-$ and the future and past timelike co-rays at p fit together to form a smooth geodesic c.

Also, we have $b^+_{p,a} \geqslant b^+ \geqslant -b^- \geqslant -b^-_{p,a}$ near p with equality at p. Since $\operatorname{grad} b^+_{p,a}(p) = -\operatorname{grad} b^-_{p,a}(p)$, it follows that B^+ and b^- are differentiable at p and $\operatorname{grad} b^+(p) = -\operatorname{grad} b^-(p) = \operatorname{grad} b^+_{p,a}(p) = -\operatorname{grad} b^-_{p,a}(p)$. Now as $v^+ = -\operatorname{grad} b^+_{p,a}(p) = -\operatorname{grad} b^+(p)$ and $v^- = -\operatorname{grad} b^-_{p,a}(p) = -\operatorname{grad} b^-(p)$ the future and past co-rays to γ at p are unique for any $p \in I(\gamma)$. This last fact then implies the continuity of $V = \operatorname{grad} b^+ = -\operatorname{grad} b^-$ since the initial tangent to future co-rays to γ varies continuously with p. Finally, since the restriction to a co-ray is a co-ray, it follows that the geodesic c formed from the union of the co-rays c^+ and c^- to γ at p is distance realizing. Since any timelike co-ray to γ has infinite length, c is also complete.

5. Splitting $I(\gamma)$ and M

We are now ready to show that $I(\gamma)$ is a metric product.

Lemma 5.1. *The set $I(\gamma)$ is isometric to a Lorentzian product $\mathbf{R} \times H$, where (H, h) is a spacelike hypersurface of $I(\gamma)$. Furthermore, each spacelike slice $\{t_0\} \times H$ corresponds to the intersection of $I(\gamma)$ with a level set of b^+ (resp., b^-).*

Proof. Fix $p \in I(\gamma)$ and let c be a geodesic with $c(0) = p$. Since $b^+_{p,a}(x) \geqslant b^+(x) = -b^-(x) \geqslant -b^-_{p,a}(x)$ for all x near p, $b^+ \circ c$ has both super support functions $b^+_{p,a} \circ c$ and subsupport functions $-b^-_{p,a} \circ c$. Corollary 4.5 shows that these support functions have arbitrarily small second derivatives for all t near 0. If L: $\mathbf{R} \to \mathbf{R}$ is any affine function, then the same is true of $b^+ \circ c - L$. It follows that $b^+ \circ c$ is an affine function near $t = 0$ and this implies $b^+ \circ c$ is an affine function for any geodesic c with image in $I(\gamma)$. Hence if $H(t_0) = \{q \in I(\gamma) \mid b^+(q) = t_0\}$ is the t_0 level set of b^+ in $I(\gamma)$, and c is any geodesic segment c: $[0, a] \to I(\gamma)$ with endpoints in $H(t_0)$, then $b^+ \circ c(0) = b^+ \circ c(a)$ implies $b^+ \circ c(t) = b^+ \circ c(0)$ for all $0 \leqslant t \leqslant a$. Thus $c([0, a]) \subset H(t_0)$, which shows $H(t_0)$ is totally geodesic.

Fixing $p \in I(\gamma)$ we let $e_1 = -\operatorname{grad} b^+(p)$, $e_2, \cdots, e_n$ be an orthonormal basis of T_pM and use this basis to obtain normal coordinates $x_1, x_2, \cdots, x_n$ near p. Any geodesic c with $c(0) = p$ has a representation as $c(t) = (t\alpha_1, \cdots, t\alpha_n)$ near p, where $\alpha_1, \cdots, \alpha_n$ are constants. The affine function $b^+ \circ c$ is given by $b^+ \circ c(t) = At + B_0$, where $B_0 = b^+(p)$. Using $\operatorname{grad} b^+ = -\partial/\partial x_1$ at p we obtain $A = \alpha_1$ and thus $b^+(x) = x_1 + b^+(p)$ in these local coordinates. This shows b^+ is smooth.

The vector field $\operatorname{grad} b^+$ is everywhere orthogonal to the totally geodesic level surfaces $H(t_0)$ and $X = \operatorname{grad} b^+$ is a unit normal field to $H(t_0)$. The second fundamental form S_X must vanish on $H(t_0)$ because this surface is totally geodesic. Thus if v, w are tangent to $H(t_0)$, $S_X(v, w) = \langle -\nabla_v X, w\rangle = 0$. On the other hand, $\langle X, X\rangle \equiv -1$ yields that $\nabla_v X$ is orthogonal to X and hence tangential to $H(t_0)$. Furthermore, X is the unit tangent to the (geodesic) co-ray to γ through each $p \in I(\gamma)$ and hence $\nabla_X X \equiv 0$. Thus $X = \operatorname{grad} b^+$ is a parallel timelike vector field on $I(\gamma)$. Hence $I(\gamma)$ splits locally isometrically by Wu's proof of the local Lorentzian de Rham Theorem (cf. [15, p. 299]).

The vector field $\operatorname{grad} b^+$ is complete since all co-rays to γ are complete geodesics which are contained in $I(\gamma)$. Consequently, the map $I(\gamma) \to I(\gamma)$ given by $p \to \exp_p(t \operatorname{grad} b^+(p))$ is an isometry of $I(\gamma)$ onto $I(\gamma)$ for each fixed $t \in \mathbf{R}$. This isometry takes level sets of b^+ to level sets of b^+. Using the induced metric H on $H(0)$ and the product metric $-dt^2 \otimes h$ on $\mathbf{R} \times H(0)$, we find that F: $\mathbf{R} \times H(0) \to I(\gamma)$ given by $F(t, p_0) = \exp_{p_0}(t \operatorname{grad} b^+(p_0))$ is an isometry onto $I(\gamma)$. This establishes the result. q.e.d.

We are finally ready to prove the main theorem.

Theorem 5.2. *Let (M, g) be a globally hyperbolic space-time of dimension $\geqslant 2$ with everywhere nonpositive timelike sectional curvatures $K \leqslant 0$ which contains a complete timelike line γ: $(-\infty, \infty) \to (M, g)$. Then (M, g) is isometric to a product $(\mathbf{R} \times H, -dt^2 \oplus h)$, where (H, h) is a complete Riemannian manifold. The factor $(\mathbf{R}, -dt^2)$ is represented by γ and (H, h) is represented by a level set of a Busemann function associated to γ.*

Proof. The set $I(\gamma)$ must be strongly causal because it is an open subset of the globally hyperbolic space-time (M, g). Furthermore, $p, q \in I(\gamma)$ implies the compact set $J^+(p) \cap J^-(q)$ also lies in $I(\gamma)$. Thus $I(\gamma)$ is globally hyperbolic. By Lemma 5.1, $I(\gamma)$ is isometric to $\mathbf{R} \times H$. But $\mathbf{R} \times H$ is globally hyperbolic implies H and $\mathbf{R} \times H$ are geodesically complete (cf. [2, p. 65]). Thus $I(\gamma)$ is geodesically complete and consequently, inextendible (cf. [2, p. 160]). Hence $I(\gamma) = M$.

Corollary 5.3. *(M, g) is geodesically complete and the level surfaces of the Busemann functions b^+ and b^- are complete (spacelike) Cauchy hypersurfaces of (M, g).*

By somewhat similar techniques, the following related results may be obtained.

Proposition 5.4. *Let (M, g) be a globally hyperbolic space-time with $\operatorname{Ric}(v, v) \geqslant 0$ on all timelike vectors v. Assume that (M, g) contains a complete timelike line γ such that every co-ray to γ is timelike and without focal points. Then (M, g) is isometric to a product $(\mathbf{R} \times H, -dt^2 \otimes h)$.*

Theorem 5.5. *Let (M, g) be a space-time with a compact Cauchy surface and everywhere nonpositive timelike sectional curvatures $K \leqslant 0$. Then either M is timelike geodesically incomplete or else M splits as in Theorem* 5.2 *with H compact.*

The proof of Theorem 5.5 uses a result of Harris [12] to show under the given hypotheses that there exists a null cut point along each future or past inextendible null geodesic. This ensures the existence of a timelike line.

References

[1] R. Bartnik, *Existence of maximal surfaces in asymptotically flat space-times*, Comm. Math. Phys. **94** (1984) 155–175.

[2] J. K. Beem & P. E. Ehrlich, *Global Lorentzian geometry*, Pure and Applied Math., Vol. 67, Dekker, New York, 1981.

[3] H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955.

[4] E. Calabi, *An extension of E. Hopf's maximum principal with an application to Riemannian geometry*, Duke Math. J. **25** (1957) 45–56.

[5] J. Cheeger & D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971) 119–128.

[6] ______, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972) 413–443.

[7] J.-H. Eschenburg & E. Heintze, *An elementary proof of the Cheeger-Gromoll splitting theorem*, Ann. Global Analysis Geometry **2** (1984) 141–151.

[8] J.-H. Eschenburg & J. J. O'Sullivan, *Jacobi tensors and Ricci curvature*, Math. Ann. **252** (1980) 1–26.

[9] G. J. Galloway, *Splitting theorems for spatially closed space-times*, Comm. Math. Phys. **96** (1984) 423–429.

[10] C. Gerhardt, *H-surfaces in Lorentzian manifolds*, Comm. Math. Phys. **89** (1983) 523–553.

[11] S. G. Harris, *A triangle comparison theorem for Lorentz manifolds*, Indiana Univ. Math. J. **31** (1982) 289–308.

[12] ______, *On maximal geodesic diameter and causality in Lorentz manifolds*, Math. Ann. **261** (1982) 307–313.

[13] S. W. Hawking & G. F. R. Ellis, *The large scale structure of space-time*, Cambridge Univ. Press, Cambridge, 1973.

[14] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, New York, 1983.

[15] H. Wu, *On the de Rham decomposition theorem*, Illinois J. Math. **8** (1964) 291–311.

[16] ______, *An elementary method in the study of nonnegative curvature*, Acta Math. **142** (1979) 57–78.

[17] S. T. Yau, ed., *Seminar on differential geometry*, Annals of Math. Studies, No. 102, Princeton Univ. Press, Princeton, NJ, 1982.

UNIVERSITY OF MISSOURI-COLUMBIA
UNIVERSITY OF MIAMI